ON FUNCTIONS OF JACOBI AND WEIERSTRASS (I)

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Abstract. We present some new results in theory of classical θ-functions of Jacobi and σ-functions of Weierstrass: ordinary differential equations (dynamical systems) and series expansions. The paper is basically organized as a stream of new formulas and constitutes, in re-casted form, some part of author's own handbook for elliptic/modular functions.

1. Introduction

Theta-functions of Jacobi and the basis of Weierstrassian functions $\sigma, \zeta, \wp, \wp'$ occur in numerous theories and applications. Since their arising, this field was a subject of intensive studies and towards the end of XIX century the state of the art in this field could be characterized about as follows. If some formula is not of particular interest then it has already been obtained. The majority of results were obtained in works of Jacobi and Weierstrass themselves and their contemporaries and followers (Hermite, Kiepert, Hurwitz, Halphen, Frobenius, Fricke, and others). There is extensive bibliography, including handbooks, on the theory of elliptic/modular and related functions. Most thorough works are four volumes by Tannery & Molk [7], two volume set by Halphen with a posthumous edition of the third volume [3] and, of course, *Werke* by Weierstrass [8] and Jacobi [6]. A large number of significant examples can be found in *"A Course of Modern Analysis"* by Whittaker.

In this work we would like to present some results concerning the θ-functions of Jacobi and σ-functions of Weierstrass. Namely, power series expansions of the functions and differential equations to be obeyed by them. In a separate work we shall expound another way of construction of the Jacobi–Weierstrass theory different from Jacobi's approach of inversion of a holomorphic elliptic integral and Weierstrass's construction of elliptic functions as doubly-periodic and meromorphic ones.

Series expansion of elliptic/modular functions is a widely exploited apparatus which, due to analytic properties, makes it possible to get exact results. It is suffice to mention the function series for various θ-quotients, the famous number-theoretic q-series, and their consequences like "Moonshine Conjecture" and McKay–Thompson series.

Proposed dynamical systems and their solutions have an independent interest because solutions of physically significant differential equations, if these are expressed through Jacobi's θ- or modular functions, are consequences of the equations presented below. First of all we should mention the theory of integrable nonlinear equations and dynamical systems. Differential properties of the "modular part" of Jacobi's functions are more transcendental and diverse. They gave rise to nice applications over the last decade known as monopoles and dynamical systems of the Halphen–Hitchin type [5], Chazy–Picard–Fuchs equations considered in works by Harnad, McKay, Ohyama and others. Our interest in this part of the problem was initiated by a problem of analytic description of moduli space of algebraic curves and differential-geometric structures on it. This problem is completely open, so that the offered "differential technique" can be useful in those situations where some classes of algebraic curves cover elliptic tori. We shall not touch this topic and restrict ourselves only to "book-keeping" part of the apparatus. It is of interest in its own right. Applications suggest themselves from the formulas.

In sections 3–10 we shall be keeping the following rule. Unless one mentioned explicitly or if nothing has been said about formula then the result is new[1]. In some cases, even though a result is obvious, consequences can be nontrivial. In particular, this concerns the differential equations on θ-functions with respect to their first argument. Differential identities sometimes occur in old literature (see for example [7]), but the differential closure is needed. In most cases the derivation of formulas is self-suggested once a given formula(s) has been displayed. For this reason we omit proofs and restrict ourselves to comments to applications or elucidations some technical details which are important for calculations.

Content of section 2 is standard and presented here to fix notation. List of bibliography, modern and classical, reduced to a minimum since, even in a shortened form, it would require mentioning tens important works.

2. Definitions and notations

2.1. **Functions of Jacobi.** Four functions $\theta_{1,2,3,4}$ and their equivalents under notation with characteristics are defined by the following series:

$$
\begin{array}{lll}
-\theta_{11}: & \theta_1(x|\tau) = -\mathrm{i} & \displaystyle\sum_{k=-\infty}^{\infty} (-1)^k \mathrm{e}^{(k+\frac12)^2\pi\mathrm{i}\tau}\,\mathrm{e}^{(2k+1)\pi\mathrm{i}x} \\
\theta_{10}: & \theta_2(x|\tau) = & \displaystyle\sum_{k=-\infty}^{\infty} \mathrm{e}^{(k+\frac12)^2\pi\mathrm{i}\tau}\,\mathrm{e}^{(2k+1)\pi\mathrm{i}x} \\
\theta_{00}: & \theta_3(x|\tau) = & \displaystyle\sum_{k=-\infty}^{\infty} \mathrm{e}^{k^2\pi\mathrm{i}\tau}\,\mathrm{e}^{2k\pi\mathrm{i}x} \\
\theta_{01}: & \theta_4(x|\tau) = & \displaystyle\sum_{k=-\infty}^{\infty} (-1)^k \mathrm{e}^{k^2\pi\mathrm{i}\tau}\,\mathrm{e}^{2k\pi\mathrm{i}x}
\end{array} .
$$

We shall use the designation: $\theta_k \equiv \theta_k(x|\tau)$. Values of θ-functions under $x = 0$ are called ϑ-constants: $\vartheta_k \equiv \vartheta_k(\tau) = \theta_k(0|\tau)$. For handy formatting formulas we shall use twofold notation for θ-functions with characteristics:

$$
\theta\!\left[{\alpha \atop \beta}\right]\!(x|\tau) = \theta_{\alpha\beta}(x|\tau) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{\pi\mathrm{i}\left(k+\frac{\alpha}{2}\right)^2\tau + 2\pi\mathrm{i}\left(k+\frac{\alpha}{2}\right)\left(x+\frac{\beta}{2}\right)}.
$$

We consider arbitrary integral characteristics and therefore the functions $\theta_{\alpha\beta}$ are always equal to $\pm\theta_{1,2,3,4}$. Shifts by $\frac12$-periods lead to shifts of characteristics:

$$
\theta\!\left[{\alpha \atop \beta}\right]\!\left(x + \tfrac{n}{2} + \tfrac{m}{2}\,\tau\big|\tau\right) = \theta\!\left[{\alpha+m \atop \beta+n}\right]\!(x|\tau)\cdot\mathrm{e}^{-\pi\mathrm{i}\,m\left(x+\frac{\beta+n}{2}+\frac{m}{4}\tau\right)}, \qquad (n,\,m) = 0, \pm1, \pm2, \ldots
$$

Twofold shifts by $\frac12$-periods yield the law of transformation of θ-function into itself:

$$
\theta_{\alpha\beta}(x + n + m\,\tau|\tau) = \theta_{\alpha\beta}(x|\tau)\cdot\mathrm{e}^{-\pi\mathrm{i}\,(m^2\tau+2mx)}\,(-1)^{n\alpha-m\beta}, \qquad (n,\,m) = 0, \pm1, \pm2, \ldots.
$$

Value of any θ-function at any $\frac12$-period is a certain ϑ-constant with an exponential multiplier:

$$
\theta\!\left[{\alpha \atop \beta}\right]\!\left(\tfrac{n}{2} + \tfrac{m}{2}\,\tau\big|\tau\right) = \vartheta\!\left[{\alpha+m \atop \beta+n}\right]\!(\tau)\cdot\mathrm{e}^{-\pi\mathrm{i}m\left(\frac{\beta+n}{2}+\frac{m}{4}\tau\right)}, \qquad (n,\,m) = 0, \pm1, \pm2, \ldots
$$

[1]This means that the formula was not found in the literature, but does not mean that it is really new one.

2.2. **Functions of Weierstrass.** Weierstrassian notation $\sigma(x|\omega,\omega')$, $\wp(x|\omega,\omega')$, ... and $\sigma(x;g_2,g_3)$, $\wp(x;g_2,g_3)$, ... is the commonly accepted. Due to known relations of homogeneity for functions σ, ζ, $\wp$, $\wp'$, the two parameters (ω,ω') or (g_2,g_3) can be replaced by one $\tau=\frac{\omega'}{\omega}$ which is called modulus of elliptic curves. We shall designate that functions as follows

$$\sigma(x|\tau)\equiv\sigma(x|1,\tau),\quad \zeta(x|\tau)\equiv\zeta(x|1,\tau)\quad \wp(x|\tau)\equiv\wp(x|1,\tau)\quad \wp'(x|\tau)\equiv\wp'(x|1,\tau)\,.$$

Weierstrassian invariants g_2, g_3 and $\Delta=g_2^3-27g_3^2$ are functions of periods or modulus and defined by known formulas of Weierstrass–Eisenstein. These series are entirely unsuited for numeric computations. Hurwitz, in his dissertation (1881), found a nice transition to function series. Such series are most effective for computations:

$$g_2(\tau)=20\,\pi^4\left\{\frac{1}{240}+\sum_{k=1}^{\infty}\frac{k^3\,\mathrm{e}^{2k\pi\mathrm{i}\tau}}{1-\mathrm{e}^{2k\pi\mathrm{i}\tau}}\right\},\qquad g_3(\tau)=\tfrac{7}{3}\,\pi^6\left\{\frac{1}{504}-\sum_{k=1}^{\infty}\frac{k^5\,\mathrm{e}^{2k\pi\mathrm{i}\tau}}{1-\mathrm{e}^{2k\pi\mathrm{i}\tau}}\right\}.$$

η-function of Weierstrass is defined by the formula $\eta(\tau)=\zeta(1|1,\tau)$ and the following formula is used for computations:

$$\eta(\tau)=2\pi^2\left\{\frac{1}{24}-\sum_{k=1}^{\infty}\frac{\mathrm{e}^{2k\pi\mathrm{i}\tau}}{\left(1-\mathrm{e}^{2k\pi\mathrm{i}\tau}\right)^2}\right\},\qquad \eta\Big(\frac{a\,\tau+b}{c\,\tau+d}\Big)=(c\tau+d)^2\,\eta(\tau)-\frac{\pi\mathrm{i}}{2}\,c\,(c\,\tau+d)\,,$$

where the numbers (a,b,c,d) are integer and $ad-bc=1$. Modular transformations are used not only in theory but also in practice since the value of modulus strongly affects the convergence of the function series. Moving τ into a fundamental domain of modular group (the process is easily automatized), one obtains the values of τ with the minimally allowable imaginary part $\Im(\tau)=\frac{\sqrt{3}}{2}$. At such "the most worst" point the series converge very fast.

Three functions of Weierstrass σ_λ are defined by the following expressions:

$$\sigma_\lambda(x|\omega,\omega')=\quad \mathrm{e}^{\eta(\omega,\omega')\,\frac{x^2}{2\omega}}\cdot\frac{\theta_{\lambda+1}\big(\frac{x}{2\omega}\big|\frac{\omega'}{\omega}\big)}{\vartheta_{\lambda+1}\big(\frac{\omega'}{\omega}\big)}\,,\qquad \text{where}\quad \lambda=1,2,3\,.$$

The σ-function of Weierstrass, as a function of $(x,\ g_2,\ g_3)$, satisfies the linear differential equations obtained by Weierstrass:

$$\left\{\begin{aligned} x\,\frac{\partial\sigma}{\partial x}-4\,g_2\,\frac{\partial\sigma}{\partial g_2}-6\,g_3\,\frac{\partial\sigma}{\partial g_3}-\sigma=0\\ \frac{\partial^2\sigma}{\partial x^2}-12\,g_3\,\frac{\partial\sigma}{\partial g_2}-\tfrac{2}{3}\,g_2^2\,\frac{\partial\sigma}{\partial g_3}+\tfrac{1}{12}\,g_2\,x^2\,\sigma=0\end{aligned}\right. . \tag{1}$$

These equations make it possible to write down the recursive relation for coefficients $C_k(g_2,g_3)$ of the power series of the function σ:

$$\sigma(x;g_2,g_3)=C_0\,x+C_1\,\frac{x^3}{3!}+\cdots=x-\tfrac{g_2}{240}\,x^5-\tfrac{g_3}{840}\,x^7+\cdots\,,\qquad \big(C_0=1,\ C_1=0\big)\,. \tag{2}$$

Two such recurrences are known. One is due to Halphen:

$$C_k=-\widehat{\mathfrak{D}}\,C_{k-1}-\tfrac{1}{6}\,(k-1)(2\,k-1)\,g_2\,C_{k-2}\,,\quad \text{where}\quad \widehat{\mathfrak{D}}=-12\,g_3\,\frac{\partial}{\partial g_2}-\tfrac{2}{3}\,g_2^2\,\frac{\partial}{\partial g_3}\,. \tag{3}$$

The second was obtained by Weierstrass:

$$\sigma(x;g_2,g_3)=\sum_{m,n=0}^{\infty}A_{m,\,n}\left(\tfrac{g_2}{2}\right)^m\left(2g_3\right)^n\frac{x^{4m+6n+1}}{(4m+6n+1)!},\qquad \left\{\begin{array}{rcl} A_{0,0}&=&1\\ A_{m,\,n}&=&0\quad\big(m<0,\ n<0\big)\end{array}\right. .$$

$$A_{m,\,n}=\frac{16}{3}\,(n+1)\,A_{m-2,\,n+1}+3\,(m+1)\,A_{m+1,\,n-1}-\frac{1}{3}\,(2m+3n-1)(4m+6\,n-1)\,A_{m-1,\,n}.$$

Recently, in connection with theory of Kleinian $\boldsymbol{\sigma}$-functions, yet another recurrence was obtained [2]. Among all the recurrences, Weierstrassian one is the least expendable. There are only multiplications of integers in it and its comparison efficiency rapidly grows under the growing of order of the expansions. Weierstrass proves separately that the numbers $A_{m,n}$ are integers.

2.3. **Function of Dedekind.** Due to coincidence the standard designations for Weierstrassian function $\eta(\tau)$ and Dedekind's one, we shall use, for the last one, the sign $\widehat{\eta}(\tau)$:

$$\widehat{\eta}(\tau) = \mathrm{e}^{\frac{\pi \mathrm{i}}{12}\tau} \prod_{k=1}^{\infty} \big(1 - \mathrm{e}^{2k\pi \mathrm{i}\tau}\big) = \mathrm{e}^{\frac{\pi \mathrm{i}}{12}\tau} \sum_{k=-\infty}^{\infty} (-1)^k \, \mathrm{e}^{(3k^2+k)\pi \mathrm{i}\tau} . \qquad \text{(Euler (1748))}$$

There is a differential relation between the two functions. It is expressed by the formula $\dfrac{1}{\widehat{\eta}}\dfrac{d\widehat{\eta}}{d\tau} = \dfrac{\mathrm{i}}{\pi}\,\eta$.

3. Differentiation of functions $\sigma, \zeta, \wp, \wp'$

Differentiations of Weierstrassian functions with respect to invariants are known [3, 7]. Rules of transformations between differentiations with respect to (g_2, g_3) and (ω, ω') are also known (Frobenius–Stickelberger (1882)) [3]. Applying them, and assuming $\Im\big(\frac{\omega'}{\omega}\big) > 0$, one obtains

$$\left\{\begin{aligned} \frac{\partial \sigma}{\partial \omega} &= -\frac{\mathrm{i}}{\pi}\Big\{\omega'\Big(\wp - \zeta^2 - \frac{1}{12}\,g_2\,x^2\Big) + 2\,\eta'\,(x\,\zeta - 1)\Big\}\,\sigma \\ \frac{\partial \sigma}{\partial \omega'} &= \frac{\mathrm{i}}{\pi}\Big\{\omega\Big(\wp - \zeta^2 - \frac{1}{12}\,g_2\,x^2\Big) + 2\,\eta\,(x\,\zeta - 1)\Big\}\,\sigma \end{aligned}\right. ,$$

$$\left\{\begin{aligned} \frac{\partial \zeta}{\partial \omega} &= -\frac{\mathrm{i}}{\pi}\,\Big\{\omega'\Big(\wp' + 2\,\zeta\,\wp - \frac{1}{6}\,g_2\,x\Big) + 2\,\eta'(\zeta - x\,\wp)\Big\} \\ \frac{\partial \zeta}{\partial \omega'} &= \frac{\mathrm{i}}{\pi}\,\Big\{\omega\,\Big(\wp' + 2\,\zeta\,\wp - \frac{1}{6}\,g_2\,x\Big) + 2\,\eta\,(\zeta - x\,\wp)\Big\} \end{aligned}\right. ,$$

$$\left\{\begin{aligned} \frac{\partial \wp}{\partial \omega} &= \frac{2\,\mathrm{i}}{\pi}\,\Big\{\omega'\Big(2\,\wp^2 + \zeta\,\wp' - \frac{1}{3}\,g_2\Big) - \eta'\big(2\,\wp + x\,\wp'\big)\Big\} \\ \frac{\partial \wp}{\partial \omega'} &= -\frac{2\,\mathrm{i}}{\pi}\,\Big\{\omega\,\Big(2\,\wp^2 + \zeta\,\wp' - \frac{1}{3}\,g_2\Big) - \eta\,\big(2\,\wp + x\,\wp'\big)\Big\} \end{aligned}\right. ,$$

$$\left\{\begin{aligned} \frac{\partial \wp'}{\partial \omega} &= \frac{\mathrm{i}}{\pi}\Big\{\omega'\big(6\,\wp\,\wp' + 12\,\zeta\,\wp^2 - g_2\,\zeta\big) - \eta'\big(6\,\wp' + 12\,x\,\wp^2 - g_2\,x\big)\Big\} \\ \frac{\partial \wp'}{\partial \omega'} &= -\frac{\mathrm{i}}{\pi}\Big\{\omega\,\big(6\,\wp\,\wp' + 12\,\zeta\,\wp^2 - g_2\,\zeta\big) - \eta\,\big(6\,\wp' + 12\,x\,\wp^2 - g_2\,x\big)\Big\} \end{aligned}\right. .$$

Setting here $(\omega = 1,\ \omega' = \tau)$ one obtains the dynamical system with a parameter x:

$$\left\{\begin{aligned} \frac{\partial \sigma}{\partial \tau} &= \frac{\mathrm{i}}{\pi}\Big\{\wp - \zeta^2 + 2\,\eta\,(x\,\zeta - 1) - \frac{1}{12}\,g_2\,x^2\Big\}\,\sigma \\ \frac{\partial \zeta}{\partial \tau} &= \frac{\mathrm{i}}{\pi}\Big\{\wp' + 2\,\eta\zeta + 2\,\wp\,(\zeta - x\,\eta) - \frac{1}{6}\,g_2\,x\Big\} \\ \frac{\partial \wp}{\partial \tau} &= -2\frac{\mathrm{i}}{\pi}\Big\{2\,\wp^2 + \wp'(\zeta - x\,\eta) - 2\,\eta\,\wp - \frac{1}{3}\,g_2\Big\} \\ \frac{\partial \wp'}{\partial \tau} &= -6\frac{\mathrm{i}}{\pi}\Big\{\wp'(\wp - \eta) + \Big(2\,\wp^2 - \frac{1}{6}\,g_2\Big)(\zeta - x\,\eta)\Big\} \end{aligned}\right. .$$

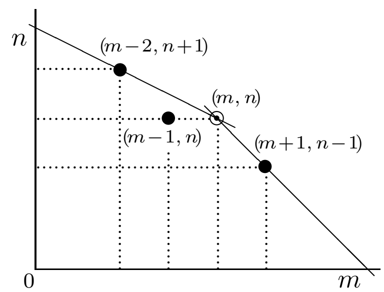
The function $\sigma(x|\tau)$ satisfies one differential equation

$$\frac{\partial^2\sigma}{\partial x^2} - 2\,x\,\eta\,\frac{\partial\sigma}{\partial x} - \pi\,\mathrm{i}\,\frac{\partial\sigma}{\partial\tau} + \Big(2\,\eta + \tfrac{1}{12}\,g_2\,x^2\Big)\,\sigma = 0\,,$$

which is an analogue of heat equation for the function $\theta(x|\tau)$. All the differential equations in variables x, ω, ω', τ contain no the quantity g_3. Depending on representation (g_2, g_3), (ω, ω') or τ, it is either algebraic integral of these equations $g_3 = 4\,\wp^3 - g_2\,\wp - \wp'^2$ or determines a fix algebraic relation between the variables. Each of the functions $\sigma, \zeta, \wp, \wp'(x|\tau)$ satisfies ordinary differential equation of 4-th order with variable coefficients $g_{2,3}(\tau)$, $\eta(\tau)$ in the variable τ. These equations have defied simplifications. By virtue of linear connection between function σ and θ, the last one also satisfies ordinary differential equations.

4. Power series for σ-functions of Weierstrass

Weierstrassian recurrence has the following graphical interpretation:



The explicitly grouped σ-series (2) has the form ($[n]$ denotes an integral part of the number)

$$\sigma(x; g_2, g_3) = \sum_{k=0}^{\infty} \left\{ \sum_{\nu=[k/3]}^{k/2} 2^{2k-5\nu} A_{3\nu-k,\,k-2\nu} \cdot g_2^{3\nu-k}\, g_3^{k-2\nu} \right\} \frac{x^{2k+1}}{(2k+1)!}\,. \tag{4}$$

Denote $e_\lambda \equiv \wp(\omega_\lambda|\omega, \omega')$. Then the functions σ_λ satisfy the equations of Halphen:

$$\left\{\begin{aligned} x\,\frac{\partial\sigma_\lambda}{\partial x} - 2\,e_\lambda\,\frac{\partial\sigma_\lambda}{\partial e_\lambda} - 4\,g_2\,\frac{\partial\sigma_\lambda}{\partial g_2} \;&=\; 0 \\ \frac{\partial^2\sigma_\lambda}{\partial x^2} - \Big(4\,e_\lambda^2 - \tfrac{2}{3}\,g_2\Big)\frac{\partial\sigma_\lambda}{\partial e_\lambda} - 12\,\big(4\,e_\lambda^3 - g_2\,e_\lambda\big)\frac{\partial\sigma_\lambda}{\partial g_2} + \Big(e_\lambda + \tfrac{1}{12}\,g_2\,x^2\Big)\sigma_\lambda \;&=\; 0 \end{aligned}\right. .$$

Herefrom one obtains analogue of the recurrence (4):

$$\sigma_\lambda(x; e_\lambda, g_2) = \sum_{k=0}^{\infty} \left\{ \sum_{\nu=0}^{k/2} 2^{-\nu}\,\mathfrak{B}_{k-2\nu,\,\nu} \cdot e_\lambda^{k-2\nu} g_2^{\nu} \right\} \frac{x^{2k}}{(2k)!}\,,$$

$$\begin{aligned} \mathfrak{B}_{m,n} = \;& 24\,(n+1)\,\mathfrak{B}_{m-3,\,n+1} + (4\,m - 12\,n - 5)\,\mathfrak{B}_{m-1,\,n} - \\ & -\frac{4}{3}\,(m+1)\,\mathfrak{B}_{m+1,\,n-1} - \frac{1}{3}(m + 2\,n - 1)(2\,m + 4\,n - 3)\,\mathfrak{B}_{m,\,n-1}\,. \end{aligned}$$

All the coefficients $\mathfrak{B}_{m,\,n}$ are integrals, and $\mathfrak{B}_{0,0} = 1$ and $\mathfrak{B}_{m,\,n} = 0$ under $(m, n) < 0$. Weierstrass wrote out recurrences for the functions $S_\lambda = \mathrm{e}^{\frac{1}{2}e_\lambda x^2}\sigma_\lambda$ in representation $\big(e_\lambda,\, \varepsilon_\lambda = 3e_\lambda^2 - \frac{1}{4}g_2\big)$.

One can build a universal series for the functions σ, σ_λ. All the functions satisfy the differential equations

$$\left\{\begin{aligned} x\,\frac{\partial\Xi}{\partial x} - 2\,e_\lambda\,\frac{\partial\Xi}{\partial e_\lambda} - 4\,g_2\,\frac{\partial\Xi}{\partial g_2} - (1-\varepsilon)\,\Xi \;=\; 0 \\ \frac{\partial^2\Xi}{\partial x^2} - \Big(4\,e_\lambda^2 - \tfrac{2}{3}\,g_2\Big)\frac{\partial\Xi}{\partial e_\lambda} - 12\,\big(4\,e_\lambda^3 - g_2\,e_\lambda\big)\frac{\partial\Xi}{\partial g_2} + \Big(\varepsilon\,e_\lambda + \tfrac{1}{12}\,g_2\,x^2\Big)\Xi \;=\; 0 \end{aligned}\right. , \tag{5}$$

where case $\Xi = \sigma_\lambda$ corresponds to $\varepsilon = 1$, and $\Xi = \sigma$ corresponds to $\varepsilon = 0$ and arbitrary e_λ. The quantity ε is a parity of θ-characteristic. The universal series and integer recurrence acquire the following form:

$$\begin{aligned} \Xi(x; e_\lambda, g_2) = {}& \sum_{k=0}^{\infty} \left\{ \sum_{\nu=0}^{k/2} 2^{-\nu}\,\mathfrak{B}^{(\varepsilon)}_{k-2\nu,\,\nu} \cdot e_\lambda^{k-2\nu} g_2^{\nu} \right\} \frac{x^{2k+1-\varepsilon}}{(2k+1-\varepsilon)!}\,, \\ \mathfrak{B}^{(\varepsilon)}_{m,n} = {}& 24\,(n+1)\,\mathfrak{B}^{(\varepsilon)}_{m-3,\,n+1} + (4\,m - 12\,n - 4 - \varepsilon)\,\mathfrak{B}^{(\varepsilon)}_{m-1,\,n} - \\ & -\frac{4}{3}\,(m+1)\,\mathfrak{B}^{(\varepsilon)}_{m+1,\,n-1} - \frac{1}{3}(m+2\,n-1)(2\,m+4\,n-1-2\,\varepsilon)\,\mathfrak{B}^{(\varepsilon)}_{m,\,n-1}\,. \end{aligned} \tag{6}$$

Transition between pairs (g_2, g_3), (e_λ, g_2) and (e_λ, e_μ) is one-to-one therefore the universal recurrence can be written for any of these representations. In the last case it is symmetrical. The representation (6) is a five-term one, in contrast to four-term recurrence of Weierstrass $A_{m,n}$, but the representation (g_2, g_3) is not possible for the functions σ_λ.

5. Power series for θ-functions of Jacobi

Jacobi made an attempt to obtain such series as early as before appearance of his *Fundamenta Nova* [6, **I**: 259–260]. These series are the series with polynomial coefficients in $\eta(\tau)$ and $\vartheta(\tau)$-constants. This fact follows from the formulas

$$\theta_1(x|\tau) = \pi\,\widehat{\eta}^3(\tau)\cdot \mathrm{e}^{-2\eta(\tau)x^2}\sigma(2x|\tau)\,, \qquad \theta_\lambda(x|\tau) = \vartheta_\lambda(\tau)\cdot \mathrm{e}^{-2\eta(\tau)x^2}\sigma_{\lambda-1}(2x|\tau)\,.$$

ϑ-constants make it possible to rewrite Weierstrassian representations g_2, g_3, e_λ into ϑ-constant ones. Formulas for branch-points e_k through the ϑ-constants are well known. In turn, ϑ-constants are connected via Jacobi identity $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$. All this enables one to convert the representations choosing arbitrary pair. We shall speak (α, β)-*representation*, if formulas are written through the constants $(\vartheta_{\alpha 0}, \vartheta_{0\beta})$ under $(\alpha, \beta) \neq (0, 0)$. Operator (3) has Weierstrassian representation

$$\begin{aligned} \widehat{\mathfrak{D}} &= -12\,\big(4\,e_\lambda^3 - g_2\,e_\lambda\big)\frac{\partial}{\partial g_2} - \Big(4\,e_\lambda^2 - \tfrac{2}{3}g_2\Big)\,\frac{\partial}{\partial e_\lambda} = \\ &= \tfrac{4}{3}\big(2\,e_\mu^2 + 2\,e_\mu e_\lambda - e_\lambda^2\big)\frac{\partial}{\partial e_\lambda} + \tfrac{4}{3}\big(2\,e_\lambda^2 + 2\,e_\lambda e_\mu - e_\mu^2\big)\frac{\partial}{\partial e_\mu}\,, \end{aligned}$$

and also ϑ-constant one. For example $(\vartheta_2, \vartheta_4)$-representation

$$\widehat{\mathfrak{D}} = \frac{\pi^2}{3}\,\big\{\vartheta_2^8 + 2\,\vartheta_2^4\,\vartheta_4^4\big\}\,\frac{\partial}{\partial\vartheta_2^4} - \frac{\pi^2}{3}\,\big\{\vartheta_4^8 + 2\,\vartheta_2^4\,\vartheta_4^4\big\}\,\frac{\partial}{\partial\vartheta_4^4}\,. \tag{7}$$

Let us denote $\langle\alpha\rangle = (-1)^\alpha$. General (α, β)-representation of Halphen's operator has the form

$$\widehat{\mathfrak{D}} = \frac{\pi^2}{3}\,\big(\langle\alpha\rangle\,\vartheta_{0\beta}^8 + 2\,\langle\beta\rangle\,\vartheta_{\alpha 0}^4\,\vartheta_{0\beta}^4\big)\frac{\partial}{\partial\vartheta_{0\beta}^4} - \frac{\pi^2}{3}\,\big(\langle\beta\rangle\,\vartheta_{\alpha 0}^8 + 2\,\langle\alpha\rangle\,\vartheta_{\alpha 0}^4\,\vartheta_{0\beta}^4\big)\frac{\partial}{\partial\vartheta_{\alpha 0}^4}\,.$$

Let ε is defined by a parity of characteristic of the function $\theta_{\alpha\beta}(x|\tau)$:

$$\varepsilon = \frac{\langle\alpha\beta\rangle + 1}{2}\,, \qquad \left\{\begin{matrix} \varepsilon = 0 & \text{under} & \theta_{\alpha\beta} = \pm\,\theta_1 \\ \varepsilon = 1 & \text{under} & \theta_{\alpha\beta} = \pm\,\theta_{2,3,4} \end{matrix}\right\}\,.$$

The equation (5) for the function $\Xi = (\sigma, \sigma_\lambda)$ has the form

$$\frac{\partial^2\,\Xi}{\partial x^2} + \widehat{\mathfrak{D}}\,\Xi + \Big\{\varepsilon\,e_\lambda(\vartheta) + \frac{\pi^4}{12^2}\big[\vartheta_2^8 + \vartheta_2^4\,\vartheta_4^4 + \vartheta_4^8\big]x^2\Big\}\,\Xi = 0\,,$$

where $\widehat{\mathfrak{D}}$ is determined by formula (7).

5.1. **Function** $\theta_1(x|\tau)$**.** Expansion of function

$$\theta_1(x|\tau) = \sum_{k=0}^{\infty} C_k(\tau)\cdot x^{2k+1} = 2\,\pi\,\widehat{\eta}^3\,\Big\{x - 2\,\eta\cdot x^3 + \Big(2\,\eta^2 - \frac{\pi^4}{180}\big(\vartheta_2^8 + \vartheta_2^4\,\vartheta_4^4 + \vartheta_4^8\big)\Big)\cdot x^5 + \cdots\Big\} \tag{8}$$

has the following analytic representation:

$$\begin{aligned}\theta_1(x|\tau) = {}& 2\,\pi\sum_{k=0}^{\infty}\frac{(4\pi\mathrm{i})^k}{(2k+1)!}\,\frac{d^k\,\widehat{\eta}^3}{d\tau^k}\cdot x^{2k+1} = \\ &= 2\,\pi\,\widehat{\eta}^3\sum_{k=0}^{\infty}(-2)^k\left\{\sum_{\nu=0}^{k}\frac{(-6)^{-\nu}\,\pi^{2\nu}}{(k-\nu)!\,(2\nu+1)!}\cdot\eta^{k-\nu}\,\mathcal{N}_\nu(\vartheta)\right\}x^{2k+1}\,,\end{aligned} \tag{9}$$

where polynomials $\mathcal{N}_\nu(\vartheta)$, depending on combinations of ϑ-constants, have the form

$$\mathcal{N}_\nu(\vartheta) = \sum_{s=0}^{\nu}\left\{\begin{matrix} \mathfrak{G}_{\nu-s,\,s}\cdot\vartheta_4^{4s}\,\vartheta_2^{4(\nu-s)} \\ (-1)^s\,\mathfrak{G}_{\nu-s,\,s}\cdot\vartheta_3^{4s}\,\vartheta_4^{4(\nu-s)} \\ (-1)^s\,\mathfrak{G}_{s,\,\nu-s}\cdot\vartheta_3^{4s}\,\vartheta_2^{4(\nu-s)} \end{matrix}\right\}.$$

Integer recurrence $\mathfrak{G}_{m,n}$ ($\mathfrak{G}_{0,\,0} = 1$) looks as follows:

$$\begin{aligned}\mathfrak{G}_{m,\,n} = {}& 4\,(n - 2\,m - 1)\,\mathfrak{G}_{m,\,n-1} - 4\,(m - 2\,n - 1)\,\mathfrak{G}_{m-1,\,n} - \\ &- 2\,(m + n - 1)(2\,m + 2\,n - 1)\big(\mathfrak{G}_{m-2,\,n} + \mathfrak{G}_{m-1,\,n-1} + \mathfrak{G}_{m,\,n-2}\big)\end{aligned}\,.$$

The recurrence $\mathfrak{G}_{m,n}$ is antisymmetric: $\mathfrak{G}_{m,n} = (-1)^{m+n}\,\mathfrak{G}_{n,m}$. One may write out representation of the type $\theta_1(x|\tau) = \sum C_{mnp}\,g_2^m\,g_3^n\,\eta^p\,x^k$ through the recurrence of Weierstrass $A_{m,n}$, but $\mathfrak{G}_{m,n}$ is more effective than $A_{m,n}$ since polynomials are already grouped together in ϑ-constants.

Odd derivatives $\theta_1^{(2k+1)}(0|\tau) = \vartheta_1^{(2k+1)}(\tau)$, i.e. expressions in front of x^{2k+1} in (9), generate polynomials (8) in (η, ϑ) which are exactly integrable k times in τ.

5.2. **Functions** $\theta_{2,3,4}(x|\tau)$**.** Expansions of functions $\theta\big[{\alpha \atop \beta}\big] = \pm\theta_{2,3,4}$ of the form

$$\theta\big[{\alpha \atop \beta}\big](x|\tau) = \sum_{k=0}^{\infty} C_k^{(\alpha,\beta)}(\tau)\cdot x^{2k} = \vartheta\big[{\alpha \atop \beta}\big] - \vartheta\big[{\alpha \atop \beta}\big]\,\Big\{2\,\eta + \frac{\pi^2}{6}\Big(\langle\beta\rangle\,\vartheta\big[{\alpha-1 \atop 0}\big]^4 - \langle\alpha\rangle\,\vartheta\big[{0 \atop \beta-1}\big]^4\Big)\Big\}\,x^2 + \cdots \tag{10}$$

have the following analytic representation:

$$\theta\big[{\alpha \atop \beta}\big](x|\tau) = \sum_{k=0}^{\infty}\frac{(4\pi\mathrm{i})^k}{(2k)!}\,\frac{d^k\vartheta\big[{\alpha \atop \beta}\big]}{d\tau^k}\cdot x^{2k}\,. \tag{11}$$

The equation (5) on functions $\Xi = (\sigma, \sigma_\lambda)$ in (α, β)-representation has the form

$$\frac{\partial^2\,\Xi}{\partial x^2} + \widehat{\mathfrak{D}}\,\Xi + \Big\{e_{\gamma\delta}(\vartheta) + \frac{\pi^4}{12^2}\big(\vartheta_{\alpha 0}^8 + \langle\alpha+\beta\rangle\,\vartheta_{\alpha 0}^4\,\vartheta_{0\beta}^4 + \vartheta_{0\beta}^8\big)x^2\Big\}\,\Xi = 0\,,$$

where $e_{\gamma\delta}(\vartheta)$ correspond (independently of representation (α,β)) to chosen function σ or σ_λ:

$$e_{\gamma\delta}(\vartheta)=\frac{\pi^2}{12}\Big(\langle\delta\rangle\,\vartheta\big[{}^{\gamma-1}_{\;0}\big]^4-\langle\gamma\rangle\,\vartheta\big[{}^{\;0}_{\delta-1}\big]^4\Big)\ ,\qquad \left\{\begin{array}{ll} e_{11}\equiv 0, & e_{10}\equiv e_1\\ e_{00}\equiv e_2, & e_{01}\equiv e_3\end{array}\right\}\ . \tag{12}$$

Representation is called symmetrical if $(\gamma,\delta)=(\alpha,\beta)$.

In symmetrical representation the series (11) has the following grouped form:

$$\theta\big[{}^{\alpha}_{\beta}\big](x|\tau)=\vartheta\big[{}^{\alpha}_{\beta}\big](\tau)\sum_{k=0}^{\infty}(-2)^k\left\{\sum_{\nu=0}^{k}\frac{(-6)^{-\nu}\,\pi^{2\nu}}{(k-\nu)!\,(2\nu)!}\cdot\eta^{k-\nu}\,\mathcal{N}_\nu^{(\alpha,\beta)}(\vartheta)\right\}x^{2k} \tag{13}$$

with the universal integer recurrence ($\mathfrak{G}_{0,\,0}^{(\alpha,\beta)}=1$):

$$\begin{aligned}
\mathcal{N}_\nu^{(\alpha,\beta)}(\vartheta)\ &=\ \sum_{s=0}^{\nu}\mathfrak{G}_{\nu-s,\,s}^{(\alpha,\beta)}\cdot\vartheta\big[{}^{\;0}_{\beta-1}\big]^{4s}\,\vartheta\big[{}^{\alpha-1}_{\;0}\big]^{4(\nu-s)}\ ,\\
\mathfrak{G}_{m,\,n}^{(\alpha,\beta)}\ &=\ \langle\alpha\rangle\,(4\,n-8\,m-3)\,\mathfrak{G}_{m,\,n-1}^{(\alpha,\beta)}-\langle\beta\rangle\,(4\,m-8\,n-3)\,\mathfrak{G}_{m-1,\,n}^{(\alpha,\beta)}-\\
&\quad-2\,(m+n-1)(2\,m+2\,n-3)\Big(\mathfrak{G}_{m-2,\,n}^{(\alpha,\beta)}+\langle\alpha+\beta\rangle\,\mathfrak{G}_{m-1,\,n-1}^{(\alpha,\beta)}+\mathfrak{G}_{m,\,n-2}^{(\alpha,\beta)}\Big)\ .
\end{aligned}$$

Symmetry of the recurrence $\mathfrak{G}_{m,\,n}^{(\alpha,\beta)}$ with respect to permutation of indices is determined by the properties:

$$\begin{aligned}
\mathfrak{G}_{n,\,m}^{(\alpha,\beta)}&=(-1)^{(m+n)(\alpha+\beta+1)}\,\mathfrak{G}_{m,\,n}^{(\alpha,\beta)}\ ,\\
\mathfrak{G}_{m,\,n}^{(\beta,\alpha)}&=(-1)^{(m+n)(\alpha+\beta)}\quad\mathfrak{G}_{m,\,n}^{(\alpha,\beta)}\ .
\end{aligned}$$

This means that there are only two recurrence $\mathfrak{G}_{m,\,n}^{(\alpha)}$ under $\alpha=\{1,0\}$ and $\beta=0$:

$$\begin{aligned}
\mathfrak{G}_{m,\,n}^{(\alpha)}\ &=\ \langle\alpha\rangle\,(4\,n-8\,m-3)\,\mathfrak{G}_{m,\,n-1}^{(\alpha)}-(4\,m-8\,n-3)\,\mathfrak{G}_{m-1,\,n}^{(\alpha)}-\\
&\quad-2\,(m+n-1)(2\,m+2\,n-3)\Big(\mathfrak{G}_{m-2,\,n}^{(\alpha)}+\langle\alpha\rangle\,\mathfrak{G}_{m-1,\,n-1}^{(\alpha)}+\mathfrak{G}_{m,\,n-2}^{(\alpha)}\Big)
\end{aligned}$$

with property $\mathfrak{G}_{n,\,m}^{(\alpha)}=(-1)^{(m+n)(\alpha+1)}\,\mathfrak{G}_{m,\,n}^{(\alpha)}$:

$$\mathfrak{G}_{n,\,m}^{(0)}=(-1)^{(m+n)}\,\mathfrak{G}_{m,\,n}^{(0)}\ ,\qquad\qquad \mathfrak{G}_{n,\,m}^{(1)}=\mathfrak{G}_{m,\,n}^{(1)}\ .$$

Developments (9) and (13) differ from each other only in a multiplier and shape of the recurrence. They can be unified into one (by introducing the parity ε) but the quantity $\langle\alpha\rangle$ is left and (m,n)-entries of matrices $\mathfrak{G}^{(\beta,\alpha)}$ differ only in a sign.

Even derivatives $\theta_{\alpha\beta}^{(2k)}(0|\tau)=\vartheta_{\alpha\beta}^{(2k)}(\tau)$, i.e. expressions in front of x^{2k} in (13), generate polynomials (10) in (η,ϑ) which are integrable k times in τ. Their integrability is a consequence of one dynamical system considered in sect. 7.

Making use of these expansions one can build other ones about points $x=\{\pm\frac{1}{2},\,\pm\frac{\tau}{2}\}$. Described series will be, up to obvious modifications, changed into one another.

6. Dynamical systems on θ-functions

6.1. **Differential equations in x.** The five functions

$$\theta_{1,2,3,4}\quad\text{and}\quad\theta_1'\equiv\frac{\partial\theta_1}{\partial x}$$

satisfy closed ordinary autonomous differential equations:

$$\left\{\begin{aligned}
\frac{\partial\theta_2}{\partial x} &= \frac{\theta_1'}{\theta_1}\,\theta_2 - \pi\,\vartheta_2^2\cdot\frac{\theta_3\theta_4}{\theta_1} \qquad & \frac{\partial\theta_1'}{\partial x} &= \frac{\theta_1'^2}{\theta_1} - \pi^2\vartheta_3^2\,\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1} - \Big\{4\eta + \frac{\pi^2}{3}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\cdot\theta_1\\
\frac{\partial\theta_3}{\partial x} &= \frac{\theta_1'}{\theta_1}\,\theta_3 - \pi\,\vartheta_3^2\cdot\frac{\theta_2\theta_4}{\theta_1} & \frac{\partial\theta_1}{\partial x} &= \theta_1'\,.\\
\frac{\partial\theta_4}{\partial x} &= \frac{\theta_1'}{\theta_1}\,\theta_4 - \pi\,\vartheta_4^2\cdot\frac{\theta_2\theta_3}{\theta_1}
\end{aligned}\right.$$

These equations are equivalent to relations between Weierstrassian functions $(\sigma,\zeta,\wp,\wp')(x|\tau)$. General form of differentiation of the functions $\theta_{1,2,3,4}$ is as follows ($\vartheta_1=0$):

$$\frac{\partial\theta_k}{\partial x} = \frac{\theta_1'}{\theta_1}\,\theta_k - \pi\,\vartheta_k^2\cdot\frac{\theta_\nu\,\theta_\mu}{\theta_1}\,, \qquad \text{where} \quad \nu=\frac{8\,k-28}{3\,k-10},\quad \mu=\frac{10\,k-28}{3\,k-8},\quad k=(1,2,3,4)\,.$$

It immediately follows that the following identities are obeyed [2]

$$\frac{\theta_n'}{\theta_n} - \frac{\theta_m'}{\theta_m} = \pi\vartheta_k^2\cdot\frac{\theta_1\theta_k}{\theta_n\theta_m}\,\mathrm{sign}(n-m)\,, \qquad \text{where} \quad (k\ne n,\ n\ne m,\ k\ne m,\quad k,n,m=2,3,4)\,,$$

and also well-known polynomial identities:

$$\theta_1^4+\theta_3^4=\theta_2^4+\theta_4^4,\quad \vartheta_3^2\,\theta_3^2=\vartheta_2^2\,\theta_2^2+\vartheta_4^2\,\theta_4^2,\quad \mathrm{sign}(n-m)\cdot\vartheta_k^2\,\theta_1^2=\vartheta_m^2\,\theta_n^2-\vartheta_n^2\,\theta_m^2\,. \tag{14}$$

6.2. **Differential equations in τ.** The functions $\theta_{1,2,3,4},\theta_1'$ satisfy closed non-autonomous ordinary differential equations in τ:

$$\left\{\begin{aligned}
\frac{\partial\theta_1}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\,\frac{\theta_1'^2}{\theta_1} + {} + \frac{\pi\mathrm{i}}{4}\,\vartheta_3^2\,\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1} + \Big\{\frac{\mathrm{i}}{\pi}\eta+\frac{\pi\mathrm{i}}{12}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\cdot\theta_1\\
\frac{\partial\theta_2}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\Big\{\frac{\theta_1'}{\theta_1}-\pi\vartheta_2^2\cdot\frac{\theta_3\theta_4}{\theta_1\theta_2}\Big\}^2\theta_2 + \frac{\pi\mathrm{i}}{4}\,\vartheta_3^2\,\vartheta_4^2\cdot\frac{\theta_1^2}{\theta_2} + \Big\{\frac{\mathrm{i}}{\pi}\eta+\frac{\pi\mathrm{i}}{12}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\cdot\theta_2\\
\frac{\partial\theta_3}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\,\frac{\theta_1'^2}{\theta_1^2}\,\theta_3 + \frac{\mathrm{i}}{2}\vartheta_3^2\cdot\theta_2\theta_4\frac{\theta_1'}{\theta_1^2} - \frac{\pi\mathrm{i}}{4}\,\vartheta_2^2\,\vartheta_3^2\cdot\frac{\theta_4^2}{\theta_1^2}\,\theta_3 + \Big\{\frac{\mathrm{i}}{\pi}\eta+\frac{\pi\mathrm{i}}{12}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\cdot\theta_3\\
\frac{\partial\theta_4}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\,\frac{\theta_1'^2}{\theta_1^2}\,\theta_4 + \frac{\mathrm{i}}{2}\vartheta_4^2\cdot\theta_2\theta_3\frac{\theta_1'}{\theta_1^2} - \frac{\pi\mathrm{i}}{4}\,\vartheta_2^2\,\vartheta_4^2\cdot\frac{\theta_3^2}{\theta_1^2}\,\theta_4 + \Big\{\frac{\mathrm{i}}{\pi}\eta+\frac{\pi\mathrm{i}}{12}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\cdot\theta_4\\
\frac{\partial\theta_1'}{\partial\tau} &= \frac{-\mathrm{i}}{4\pi}\,\frac{\theta_1'^3}{\theta_1^2} + 3\Big\{\frac{\pi\mathrm{i}}{4}\vartheta_3^2\,\vartheta_4^2\cdot\frac{\theta_2^2}{\theta_1^2}+\frac{\mathrm{i}}{\pi}\eta+\frac{\pi\mathrm{i}}{12}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\,\theta_1' - \frac{\pi^2}{2}\mathrm{i}\,\vartheta_2^2\,\vartheta_3^2\,\vartheta_4^2\cdot\frac{\theta_2\theta_3\theta_4}{\theta_1^2}
\end{aligned}\right. . \tag{15}$$

General form of τ-differentiation of the functions $\theta_{1,2,3,4}$ is as follows:

$$\begin{aligned}
\frac{\partial\theta_k}{\partial\tau} \;=\; & \frac{-\mathrm{i}}{4\pi}\,\frac{\theta_1'^{\,2}}{\theta_1^2}\,\theta_k + \frac{\mathrm{i}}{2}\,\vartheta_k^2\cdot\theta_\nu\,\theta_\mu\frac{\theta_1'}{\theta_1^2} + \frac{\pi\mathrm{i}}{4}\left\{\vartheta_3^2\,\vartheta_4^2\cdot\theta_2^2 - \vartheta_k^2\,\vartheta_\nu^2\,\vartheta_\mu^2\cdot\left(\frac{\theta_\nu^2}{\vartheta_\nu^2}+\frac{\theta_\mu^2}{\vartheta_\mu^2}\right)\right\}\theta_k + {}\\
& + \Big\{\frac{\mathrm{i}}{\pi}\,\eta+\frac{\pi\mathrm{i}}{12}\big(\vartheta_3^4+\vartheta_4^4\big)\Big\}\cdot\theta_k\,, \qquad \text{where} \quad \nu=\frac{8\,k-28}{3\,k-10},\quad \mu=\frac{10\,k-28}{3\,k-8},\quad k=(1,2,3,4)\,.
\end{aligned}$$

[2] These fundamental relations appear implicitly in works by Jacobi but have not got into comprehensive handbook for elliptic functions compiled by Schwarz on the basis of Weierstrass's lectures.

7. Differential equations on ϑ, η-constants

Differential closure is provided by the constants $\vartheta_{2,3,4}$ and η:

$$
\begin{aligned}
&\frac{1}{\vartheta_2}\frac{d\vartheta_2}{d\tau} = \frac{\mathrm{i}}{\pi}\,\eta + \frac{\pi\,\mathrm{i}}{12}\left(\vartheta_3^4+\vartheta_4^4\right), \qquad &&\frac{1}{\vartheta_4}\frac{d\vartheta_4}{d\tau} = \frac{\mathrm{i}}{\pi}\,\eta - \frac{\pi\,\mathrm{i}}{12}\left(\vartheta_2^4+\vartheta_3^4\right),\\
&\frac{1}{\vartheta_3}\frac{d\vartheta_3}{d\tau} = \frac{\mathrm{i}}{\pi}\,\eta + \frac{\pi\,\mathrm{i}}{12}\left(\vartheta_2^4-\vartheta_4^4\right), &&\frac{d\eta}{d\tau} = \frac{\mathrm{i}}{\pi}\Big\{2\,\eta^2 - \frac{\pi^4}{12^2}\left(\vartheta_2^8+\vartheta_3^8+\vartheta_4^8\right)\Big\}.
\end{aligned}
\tag{16}
$$

Well known differential equations on logarithms of ratios $\vartheta_2 : \vartheta_3 : \vartheta_4$ and also known dynamical system of Halphen and its numerous varieties are widely encountered in the modern literature. These are generated by the system of equations

$$
\frac{dg_2}{d\tau} = \frac{\mathrm{i}}{\pi}\Big(8\,g_2\,\eta - 12\,g_3\Big), \qquad \frac{dg_3}{d\tau} = \frac{\mathrm{i}}{\pi}\Big(12\,g_3\,\eta - \frac{2}{3}\,g_2^2\Big), \qquad \frac{d\eta}{d\tau} = \frac{\mathrm{i}}{\pi}\Big(2\,\eta^2 - \frac{1}{6}\,g_2\Big),
$$

which, in implicit form, was written out by Weierstrass and Halphen used it [3, **I:** p. 331] in order to get his famous symmetrical version of the system. Ramanujan obtained its equivalent for some number-theoretic q-series. See for example [9, § 1] and additional information in this work. It is lesser known, to all appearances, that another version of the system was written out by Jacobi in connection with power series of θ-functions. These results were published by Borchardt on the basis of papers kept after Jacobi [6, **II**: 383–398].

Jacobi obtained a dynamical system on four functions (A, B, a, b) which are, in our notation, rational functions of the ϑ, η-constants

$$
A = \vartheta_3^2\,, \quad B = \frac{4}{\pi^2}\,\frac{\eta}{\vartheta_3^2} + \frac{1}{3}\,\frac{\vartheta_2^4-\vartheta_4^4}{\vartheta_3^2}\,, \quad a = 4\left(1 - 2\,\frac{\vartheta_2^2}{\vartheta_3^2}\right), \quad b = 2\,\frac{\vartheta_2^2\vartheta_4^2}{\vartheta_3^4}
$$

and showed that the dynamical system can be useful for obtaining power series of θ-functions. He also presented two (canonical) transformations of the variables (A, B, a, b) retaining the shape of the equations. Jacobi noticed that the developments become simple and have a recursive form when extracted an exponential multiplier $\mathrm{e}^{-\frac{1}{2}ABx^2}$. This is, in fact, Halphen's recurrence (3) for the σ-function of Weierstrass. Transition between variables in mentioned systems is not one-to-one but always algebraic. This is because all of them are consequences of the equations (16) since the variables, appeared in these systems, are rationally expressible through the ϑ, η-constants. A related dynamical system arose in a work of Jacobi earlier, when he derived his known equation on ϑ-constants [6, **II:** p. 176]

$$
\left(\vartheta^2\,\vartheta_{\tau\tau\tau} - 15\,\vartheta\,\vartheta_\tau\vartheta_{\tau\tau} + 30\,\vartheta_\tau^3\right)^2 + 32\left(\vartheta\,\vartheta_{\tau\tau} - 3\,\vartheta_\tau^2\right)^3 = -\pi^2\,\vartheta^{10}\left(\vartheta\,\vartheta_{\tau\tau} - 3\,\vartheta_\tau^2\right)^2.
$$

Logarithmic derivatives of ϑ-constants, in turn, satisfy compact differential equation

$$
\left(f_\tau - 2\,f^2\right)f_{\tau\tau\tau} - f_{\tau\tau}^2 + 16\,f^3 f_{\tau\tau} + 4\,f_\tau^2\left(f_\tau - 6\,f^2\right) = 0, \qquad f = \frac{d}{d\tau}\ln\vartheta_{2,3,4}(\tau)
$$

and Dedekind's function $\Lambda = \ln\widehat{\eta}(\tau)$ satisfies the differential equation of the 3-rd order

$$
\left\{\Lambda_{\tau\tau\tau} - 12\,\Lambda_{\tau\tau}\,\Lambda_\tau + 16\,\Lambda_\tau^3\right\}^2 + 32\left\{\Lambda_{\tau\tau} - 2\,\Lambda_\tau^2\right\}^3 = \frac{4}{27}\,\pi^6\,\widehat{\eta}^{24}\,.
$$

8. Modular transformations of functions θ_1 and $\widehat{\eta}$

The modular transformation is necessary for computations of $\theta, \widehat{\eta}$-functions. General $\mathrm{PSL}_2(\mathbb{Z})$-transformation for the function θ_1 is closed in itself (in contrast to $\theta_{2,3,4}$):

$$\left\{\begin{array}{l} \theta_1\Big(\dfrac{x}{c\,\tau+d}\Big|\dfrac{a\,\tau+b}{c\,\tau+d}\Big) = \boldsymbol{\varepsilon}_\theta(a,c,d)\cdot\sqrt{c\,\tau+d}\;\,\mathrm{e}^{\frac{\pi\mathrm{i}\,c\,x^2}{c\,\tau+d}}\,\theta_1(x|\tau), \qquad \boldsymbol{\varepsilon}_\theta(a,c,d)=\sqrt[8]{1} \\ \\ \boldsymbol{\varepsilon}_\theta(a,c,d) = \mathrm{e}^{3\pi\mathrm{i}\left\{\frac{a-d}{12\,c}-\frac{d}{6}(2c-3)-\frac{1}{4}-\frac{c-1}{4}\mathrm{sign}(-d)+\frac{1}{c}\sum\limits_{k=1}^{c-1}k\left[\frac{d}{c}k\right]\right\}}, \qquad \text{under normalization} \quad c>0 \\ \\ \theta_1(x|\tau+N) = \mathrm{e}^{\frac{\pi\mathrm{i}}{4}N}\cdot\theta_1(x|\tau) \end{array}\right.$$

When discussing the modular properties of θ-functions one is usually pointed out "where $\boldsymbol{\varepsilon}_\theta$ is an eighth root of unity" but complete algorithm of its computation leaves aside. Meantime, as it follows from the shape of θ-series, it is seen that the series, which can be transformed into a "hyper-convergent" form, can turn out to be incomputable. It is known that the multiplier $\boldsymbol{\varepsilon}_\theta(a,c,d)$ can be expressed via Jacobi's symbol $\left(\frac{a}{b}\right)$ (Hermite). It has independent rules of computations. These formulas are displayed here for completeness of computations and contain some simplifications of Dedekind's sums (see [1] about them):

$$\left\{\begin{array}{rcl} \widehat{\eta}\Big(\dfrac{a\,\tau+b}{c\,\tau+d}\Big) &=& \mathrm{e}^{\pi\mathrm{i}\left\{\frac{a-d}{12\,c}-\frac{d}{6}(2c-3)-\frac{1}{4}-\frac{c-1}{4}\mathrm{sign}(-d)+\frac{1}{c}\sum\limits_{k=1}^{c-1}k\left[\frac{d}{c}k\right]\right\}}\sqrt{c\,\tau+d}\;\,\widehat{\eta}(\tau) \\ \\ \widehat{\eta}(\tau+N) &=& \mathrm{e}^{\frac{\pi\mathrm{i}}{12}N}\,\widehat{\eta}(\tau) \end{array}\right. .$$

9. θ-functions with characteristics

The preceding results, making use of notation with characteristics, allow us to unify formulas and can be a subject for further generalizations to higher genera.

9.1. **(α,β)-representations.** Any object, symmetrical in ϑ-constants, can be rewritten in (α,β)-representation. For example representation of branch points through the ϑ-constants (12). $\vartheta_{\alpha\beta}$-representations for quantities $g_{2,3}$ have the following form:

$$g_2(\tau) = \frac{\pi^4}{12}\left\{\vartheta_{\alpha0}^8 + \langle\alpha+\beta\rangle\,\vartheta_{\alpha0}^4\,\vartheta_{0\beta}^4 + \vartheta_{0\beta}^8\right\}, \qquad (\alpha,\beta)\neq(0,0)$$

$$g_3(\tau) = \frac{\pi^6}{432}\Big\{2\,\langle\beta\rangle\,\vartheta_{\alpha0}^{12} - 3\,\vartheta_{\alpha0}^4\,\vartheta_{0\beta}^4\big(\langle\beta\rangle\,\vartheta_{0\beta}^4 - \langle\alpha\rangle\,\vartheta_{\alpha0}^4\big) - 2\,\langle\alpha\rangle\,\vartheta_{0\beta}^{12}\Big\}.$$

Identity $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$ and Jacobi's formula $\vartheta_1' = 2\pi\widehat{\eta}^3$, in (α,β)-representation, have the form

$$\begin{aligned} \vartheta\big[{}^{\alpha}_{\beta}\big]^4 &= \Big(\langle\beta\rangle\,\vartheta\big[{}^{\alpha-1}_{0}\big]^4 + \langle\alpha\rangle\,\vartheta\big[{}^{0}_{\beta-1}\big]^4\Big)\,\frac{\langle\alpha\beta\rangle+1}{2} \\ \vartheta_{\alpha\beta}'(\tau) &= \mathrm{i}^\beta\,(1-\langle\alpha\beta\rangle)\cdot\pi\,\mathrm{i}\,\widehat{\eta}^3(\tau), \qquad \text{where} \quad \vartheta_{\alpha\beta}'(\tau)\equiv\theta_{\alpha\beta}'(0|\tau). \end{aligned} \tag{17}$$

In general case, when the ϑ'-constant is a value of the derivative of θ-function at some $\frac{1}{2}$-period, the formula of Jacobi (17) is generalized to the following expression:

$$\theta_{\alpha\beta}'\Big(\frac{n}{2}+\frac{m}{2}\,\tau\Big|\tau\Big) = (-\mathrm{i})^{m(\beta+n)}\cdot\pi\,\mathrm{i}\,\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}m^2\tau}\,\Big\{\mathrm{i}^{\beta+n}\,\big(1-\langle\alpha+m\rangle^{\beta+n}\big)\cdot\widehat{\eta}^3 - m\,\vartheta\big[{}^{\alpha+m}_{\beta+n}\big]\Big\}. \tag{18}$$

Algebraic and differential closeness of Jacobi's functions with integral characteristics lead to the fact that all the derivatives $\theta'_{\alpha\beta}(x|\tau)$, with arguments shifted by arbitrary $\frac{1}{2}$-periods, are expressible in terms of the function $\theta'_1(x|\tau)$ and functions $\theta_{\alpha\beta}(x|\tau)$:

$$\theta'_{\alpha\beta}\Big(x+\frac{n}{2}+\frac{m}{2}\,\tau\Big|\tau\Big) = (-\mathrm{i})^{m(\beta+n)}\cdot \mathrm{e}^{-\pi\mathrm{i}m\left(x+\frac{m}{4}\tau\right)}\left\{\Big(\frac{\theta'_1(x|\tau)}{\theta_1(x|\tau)}-\pi\mathrm{i}\,m\Big)\,\theta\big[{\alpha+m \atop \beta+n}\big](x|\tau)-\right.$$
$$\left.-\,\langle\alpha+m\rangle^{\left[\frac{\beta+n}{2}\right]}\cdot\pi\,\vartheta\big[{\alpha+m \atop \beta+n}\big]^2\cdot\frac{\theta\big[{1-\alpha-m \atop 0}\big](x|\tau)\,\vartheta\big[{0 \atop 1-\beta-n}\big](x|\tau)}{\theta_1(x|\tau)}\right\}.$$

Under $x=0$ need to use the formula (18).

9.2. **Modular transformations.** The modular transformations served as the key for Hermite when he obtained his celebrated solution of the quintic equation $x^5-x-A=0$[3]. Hermite wrote out the multiplier $\boldsymbol{\varepsilon}_\theta(a,c,d)$ in a form of Gaussian sum of exponents. We propose the exponent of a sum what is simpler. General modular transformation $\tau\to\frac{a\,\tau+b}{c\,\tau+d}$ for the functions θ has the form

$$\begin{cases}\theta\big[{\alpha'-1 \atop \beta'-1}\big]\Big(\dfrac{x}{c\,\tau+d}\Big|\dfrac{a\,\tau+b}{c\,\tau+d}\Big)=\boldsymbol{\varepsilon}_\theta(a,c,d)\,\mathrm{e}^{\frac{\pi\mathrm{i}}{4}\left\{2\alpha(bc\,\beta-d+1)-c\beta(a\beta-2)-db\,\alpha^2\right\}}\cdot\sqrt{c\,\tau+d}\;\;\mathrm{e}^{\frac{\pi\mathrm{i}\,cx^2}{c\,\tau+d}}\;\theta\big[{\alpha-1 \atop \beta-1}\big](x|\tau)\\[2ex] \boldsymbol{\varepsilon}_\theta(a,c,d)=\mathrm{e}^{3\pi\mathrm{i}\left\{\frac{a-d}{12\,c}-\frac{d}{6}(2c-3)-\frac{1}{4}-\frac{c-1}{4}\mathrm{sign}(-d)+\frac{1}{c}\sum\limits_{k=1}^{c-1}k\left[\frac{d}{c}k\right]\right\}},\qquad \text{under normalization}\quad c>0\\[2ex] \theta\big[{\alpha-1 \atop \beta}\big](x|\tau+N)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}N(\alpha^2-1)}\cdot\theta\big[{\alpha-1 \atop \beta+N\alpha}\big](x|\tau)\end{cases}$$

where, at given modular transformation, characteristics are computed by the formulas:

$$\begin{cases}\alpha'=d\,\alpha-c\,\beta\\ \beta'=-b\,\alpha+a\,\beta\end{cases},\qquad\qquad \begin{cases}\alpha=a\,\alpha'+c\,\beta'\\ \beta=b\,\alpha'+d\,\beta'\end{cases}.$$

Under $x=0$ the transformation turns into the transformation of the ϑ-constants

$$\begin{cases}\vartheta\big[{\alpha'-1 \atop \beta'-1}\big]\Big(\dfrac{a\,\tau+b}{c\,\tau+d}\Big)=\boldsymbol{\varepsilon}_\theta(a,c,d)\,\mathrm{e}^{\frac{\pi\mathrm{i}}{4}\left\{2\alpha(bc\,\beta-d+1)-c\beta(a\beta-2)-db\,\alpha^2\right\}}\cdot\sqrt{c\,\tau+d}\;\vartheta\big[{\alpha-1 \atop \beta-1}\big](\tau)\\[2ex] \boldsymbol{\varepsilon}_\theta(a,c,d)=\mathrm{e}^{3\pi\mathrm{i}\left\{\frac{a-d}{12\,c}-\frac{d}{6}(2c-3)-\frac{1}{4}-\frac{c-1}{4}\mathrm{sign}(-d)+\frac{1}{c}\sum\limits_{k=1}^{c-1}k\left[\frac{d}{c}k\right]\right\}},\qquad \text{under normalization}\quad c>0\\[2ex] \vartheta\big[{\alpha-1 \atop \beta}\big](\tau+N)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}N(\alpha^2-1)}\cdot\vartheta\big[{\alpha-1 \atop \beta+N\alpha}\big](\tau)\end{cases}$$

If characteristics $(\alpha,\beta)=\{0,1\}$ mod 2 then the formulas are closed: $(\alpha',\beta')=\{0,1\}$ mod 2. Ratio of any two ϑ,θ-functions contains no the multiplier $\boldsymbol{\varepsilon}_\theta(a,c,d)$. Hermite used this fact to build his functions $\varphi,\psi,\chi(\tau)$ and transformations between them.

[3]Solution of this equation is expressed, in fact, in terms of ϑ-constants and was obtained by Hermite in 1858 (see last formula on p. 10 in [4]). Curiously that it contains an erroneous sign which is repeated, to the best of our knowledge, everywhere the solution is reproduced.

9.3. **Multiplication theorems.** Let n be arbitrary complex number. Then the functions θ satisfy the complex multiplication theorems determined by the following recurrences :

$$\left\{\begin{aligned}
&\theta_1(nx)=\frac{\theta_3^2(n_1x)\,\theta_2^2(x)-\theta_2^2(n_1x)\,\theta_3^2(x)}{\vartheta_4^2\cdot\theta_1\big((n-2)x\big)}\,, \qquad \theta_1(2x)=2\,\theta_1(x)\,\frac{\theta_2(x)\,\theta_3(x)\,\theta_4(x)}{\vartheta_2\vartheta_3\vartheta_4}\\
&\theta\big[{\alpha+1\atop\beta+1}\big](nx)=-\frac{\langle\beta\rangle\,\theta\big[{\alpha\atop 0}\big]^2(n_1x)\,\theta\big[{\alpha\atop 0}\big]^2(x)+\langle\alpha\rangle\,\theta\big[{0\atop\beta}\big]^2(n_1x)\,\theta\big[{0\atop\beta}\big]^2(x)}{\vartheta\big[{\alpha+1\atop\beta+1}\big]^2\cdot\theta\big[{\alpha+1\atop\beta+1}\big]\big((n-2)x\big)}\,, \qquad n_1\equiv n-1
\end{aligned}\right. .$$

This is not the only representation. Multiplication for θ_1' is derived by taking a derivative. If n is integer then the formulas are closed. In particular, multiplications for the functions $\theta_{2,3,4}$ are closed (doublings were written out by Jacobi):

$$\left\{\begin{aligned}
\theta_2(nx)&=\frac{\theta_3^2(n_1x)\,\theta_3^2(x)-\theta_4^2(n_1x)\,\theta_4^2(x)}{\vartheta_2^2\cdot\theta_2\big((n-2)x\big)}\\
\theta_3(nx)&=\frac{\theta_2^2(n_1x)\,\theta_2^2(x)+\theta_4^2(n_1x)\,\theta_4^2(x)}{\vartheta_3^2\cdot\theta_3\big((n-2)x\big)}\\
\theta_4(nx)&=\frac{\theta_3^2(n_1x)\,\theta_3^2(x)-\theta_2^2(n_1x)\,\theta_2^2(x)}{\vartheta_4^2\cdot\theta_4\big((n-2)x\big)}
\end{aligned}\right. .$$

There exist the polynomial non-recursive multiplications but these are multiplications not over the constants $\vartheta_{\alpha\beta}(\tau)$. This follows from Weierstrassian formulas for the functions $\sigma,\sigma_\lambda(nx)$.

9.4. Differential equations.

9.4.1. *θ-functions.* θ-functions with arbitrary integer characteristics (α,β), as functions of the two variables x and τ, satisfy splitted and closed systems of ordinary differential equations over the differential field of ϑ,η-constants $\mathbb{C}_\partial(\vartheta,\eta)$:

$$\left\{\begin{aligned}
\frac{\partial\theta\big[{\alpha\atop\beta}\big]}{\partial x}&=\frac{\theta_1'}{\theta_1}\,\theta\big[{\alpha\atop\beta}\big]-\langle\alpha\rangle^{\left[\frac{\beta}{2}\right]}\pi\,\vartheta\big[{\alpha\atop\beta}\big]^2\cdot\frac{\theta\big[{1-\alpha\atop 0}\big]\,\theta\big[{0\atop 1-\beta}\big]}{\theta_1}\\
\frac{\partial\theta_1'}{\partial x}&=\frac{\theta_1'^2}{\theta\big[{1\atop 1}\big]}-\pi^2\vartheta\big[{0\atop 0}\big]^2\,\vartheta\big[{0\atop 1}\big]^2\cdot\frac{\theta\big[{1\atop 0}\big]^2}{\theta\big[{1\atop 1}\big]}-\left\{4\eta+\frac{\pi^2}{3}\Big(\vartheta\big[{0\atop 0}\big]^4+\vartheta\big[{0\atop 1}\big]^4\Big)\right\}\cdot\theta\big[{1\atop 1}\big]
\end{aligned}\right. , \tag{19}$$

$$\left\{\begin{aligned}
\frac{\partial\theta\big[{\alpha\atop\beta}\big]}{\partial\tau}&=\frac{-\mathrm{i}}{4\pi}\,\frac{\theta_1'^{\,2}}{\theta\big[{1\atop 1}\big]^2}\,\theta\big[{\alpha\atop\beta}\big]+\langle\alpha\rangle^{\left[\frac{\beta}{2}\right]}\,\frac{\mathrm{i}}{2}\,\vartheta\big[{\alpha\atop\beta}\big]^2\cdot\frac{\theta\big[{1-\alpha\atop 0}\big]\,\theta\big[{0\atop 1-\beta}\big]}{\theta\big[{1\atop 1}\big]^2}\,\theta_1'\,+\\
&\quad+\frac{\pi\mathrm{i}}{4}\,\vartheta\big[{1\atop 0}\big]^2\,\vartheta\big[{0\atop 0}\big]^2\,\vartheta\big[{0\atop 1}\big]^2\cdot\left\{\frac{\theta\big[{1\atop 0}\big]^2}{\vartheta\big[{1\atop 0}\big]^2}-\frac{1}{2}\,(1+\langle\alpha\beta\rangle)\cdot\left(\frac{\theta\big[{1-\alpha\atop 0}\big]^2}{\vartheta\big[{1-\alpha\atop 0}\big]^2}+\frac{\theta\big[{0\atop 1-\beta}\big]^2}{\vartheta\big[{0\atop 1-\beta}\big]^2}\right)\right\}\theta\big[{\alpha\atop\beta}\big]\,+\\
&\quad+\left\{\frac{\mathrm{i}}{\pi}\,\eta+\frac{\pi\mathrm{i}}{12}\Big(\vartheta\big[{0\atop 0}\big]^4+\vartheta\big[{0\atop 1}\big]^4\Big)\right\}\cdot\theta\big[{\alpha\atop\beta}\big]\\
\frac{\partial\theta_1'}{\partial\tau}&=\frac{-\mathrm{i}}{4\pi}\,\frac{\theta_1'^3}{\theta\big[{1\atop 1}\big]^2}+3\left\{\frac{\pi\mathrm{i}}{4}\vartheta\big[{0\atop 0}\big]^2\,\vartheta\big[{1\atop 0}\big]^2\cdot\frac{\theta\big[{0\atop 1}\big]^2}{\theta\big[{1\atop 1}\big]^2}+\frac{\mathrm{i}}{\pi}\eta+\frac{\pi\mathrm{i}}{12}\Big(\vartheta\big[{0\atop 0}\big]^4+\vartheta\big[{0\atop 1}\big]^4\Big)\right\}\theta_1'\,-\\
&\quad-\mathrm{i}\,\frac{\pi^2}{2}\,\vartheta\big[{1\atop 0}\big]^2\,\vartheta\big[{0\atop 0}\big]^2\,\vartheta\big[{0\atop 1}\big]^2\cdot\frac{\theta\big[{1\atop 0}\big]\theta\big[{0\atop 0}\big]\theta\big[{0\atop 1}\big]}{\theta\big[{1\atop 1}\big]^2}
\end{aligned}\right. . \tag{20}$$

These differentiations are not quite symmetrical. In all likelihood, the separability of variables in differential properties of the θ's is not accidental. Especially in those cases, where exact solutions of integrable nonlinear partial differential equations are expressible through the θ-functions. All such solutions, essentially non-stationary and multi-phase, are consequences of integrability of the only system of ordinary differential equations on the five Jacobi's functions θ, θ_1'. They satisfy the heat equation but it is a partial differential equation.

General solution of dynamical system (19) is given by the formulas

$$\theta_{\alpha\beta} = a \cdot \theta_{\alpha\beta}\big(x+b|\tau\big)\, \mathrm{e}^{cx}\,, \quad \theta_1' = a \cdot \big\{\theta_1'(x+b|\tau\big) - c\,\theta_{11}(x+b|\tau)\big\}\, \mathrm{e}^{cx}$$

with arbitrary functions $a, b, c(\tau)$. The system (20) has a solution of the form

$$\theta_{\alpha\beta} = a \cdot \theta_{\alpha\beta}(b|\tau)\,, \quad \theta_1' = -a' \cdot \theta_{11}\big(b|\tau\big) + a \cdot \theta_1'(b|\tau)\,,$$

with arbitrary functions $a, b(x)$. All the solutions should be supplemented by the relations (14).

9.4.2. *ϑ-constants.* (α,β)-representation for the system of equations (16) has the following form

$$\left\{\begin{aligned} \frac{d\vartheta\big[{}^{\alpha}_{\beta}\big]}{d\tau} &= \vartheta\big[{}^{\alpha}_{\beta}\big]\left\{\frac{\mathrm{i}}{\pi}\,\eta + \frac{\pi \mathrm{i}}{12}\Big(\langle\beta\rangle\,\vartheta\big[{}^{1-\alpha}_{0}\big]^4 - \langle\alpha\rangle\,\vartheta\big[{}^{0}_{1-\beta}\big]^4\Big)\right\} \\ \frac{d\eta}{d\tau} &= \frac{\mathrm{i}}{\pi}\left\{2\,\eta^2 - \frac{\pi^4}{72}\Big(\vartheta\big[{}^{\alpha}_{0}\big]^8 + \vartheta\big[{}^{0}_{\beta}\big]^8 + \langle\alpha+\beta\rangle\,\vartheta\big[{}^{\alpha}_{0}\big]^4\,\vartheta\big[{}^{0}_{\beta}\big]^4\Big)\right\} \qquad \leftarrow \qquad (\alpha,\beta) \neq (0,0)\,. \end{aligned}\right. \tag{21}$$

Its general solution, with three arbitrary constants $(a:b:c:d)$, is given by the formulas

$$\vartheta\big[{}^{\alpha}_{\beta}\big] = \sqrt[-2]{c\tau+d}\cdot\vartheta\big[{}^{\alpha}_{\beta}\big]\Big(\frac{a\tau+b}{c\tau+d}\Big)\,, \qquad \eta = (c\tau+d)^{-2}\cdot\eta\Big(\frac{a\tau+b}{c\tau+d}\Big) + \frac{1}{2}\frac{\pi \mathrm{i} c}{c\tau+d}$$

and the first identity of Jacobi (17). Right hand sides in these formulas are the ϑ, η-series. Unless one assumes that the quantities η, ϑ in equations (20) being the $\eta(\tau), \vartheta(\tau)$-constants, then the requirement of differential closeness of the field of coefficients of the system, due to heat equation, leads to the equations (21). One may also view the equations (21) as the integrability condition (compatibility) of the two systems of equations (19) and (20).

10. Conclusion

Dynamical systems of Jacobi–Halphen, their generalizations, known as $SU(2)$-invariant self-dual Einstein's equations [5], modular solutions of equations of Painlevé $\mathrm{P_{VI}}$ (Picard, Okamoto, Manin, and others) etc are also consequences of splittability of the equations on θ-functions. Thus, rules of differentiations (19–21) generate mentioned dynamical systems and their solutions and the quantities θ, η, ϑ are the uniformizing variables for them. Higher derivatives of the θ-functions or ϑ-constants are again the $\eta, \vartheta, \theta, \theta_1'$-functions in form of rational polynomials of them. Using these facts one can find new solutions or simplify known ones. For example solution of equation $\mathrm{P_{VI}}$ obtained by Hitchin in the work [5].

Modular transformations of sect. 9.2 generate automorphisms of the systems (19–21). For the ϑ, θ-functions these are merely linear transformations. For other dynamical systems, like Jacobi's equations on (A, B, a, b), this yields non-obvious transformations of dynamical variables and systems into themselves. For the modular solutions of equations of the Painlevé type the transformations become lesser obvious and cease to be canonical and change shape of the equations.

In conclusion we note that it is of interest to generalize preceding results to Θ-functions of higher genera and their constants. In some particular cases this problem might be tested in the following manner. Let us consider a situation when the Jacobian variety admits a decomposition of two-dimensional Θ-function to functions of Jacobi. For example

$$\Theta\Big(\begin{matrix} z_1 \\ z_2\end{matrix}\Big|\begin{matrix}\tau & \frac{1}{2} \\ \frac{1}{2} & \mu\end{matrix}\Big) = \frac{1}{2}\,\big(\theta_3(z_1|\tau) + \theta_4(z_1|\tau)\big)\,\theta_3\big(z_2|\mu\big) + \frac{1}{2}\,\big(\theta_3(z_1|\tau) - \theta_4(z_1|\tau)\big)\,\theta_4\big(z_2|\mu\big)\,.$$

Corresponding ten $\boldsymbol{\vartheta}\big[{\alpha \atop \beta}\big]$-constants are expressible through the quantities

$$\vartheta_{2,3,4}(\tau)\,, \quad \vartheta_{2,3,4}(\mu)\,, \quad \theta_{1,2,3,4}\big(\tfrac{1}{4}\big|\tau\big)\,, \quad \theta_{1,2,3,4}\big(\tfrac{1}{4}\big|\mu\big)\,.$$

Making use of the relations

$$2\,\theta_3^4\big(\tfrac{1}{4}\big) = \vartheta_4\vartheta_3^3 + \vartheta_3\vartheta_4^3\,, \qquad 2\,\theta_2^4\big(\tfrac{1}{4}\big) = \vartheta_4\vartheta_3^3 - \vartheta_3\vartheta_4^3\,,$$

$$\theta_4\big(\tfrac{1}{4}\big) = \theta_3\big(\tfrac{1}{4}\big)\,, \qquad \theta_1\big(\tfrac{1}{4}\big) = \frac{1}{2}\,\frac{\vartheta_2^2\,\vartheta_3\,\vartheta_4}{\theta_2\big(\tfrac{1}{4}\big)\,\theta_3^2\big(\tfrac{1}{4}\big)}$$

(these are obtained from the multiplication theorems) one may analyze the system of derivatives of the $\boldsymbol{\vartheta}$-constants with respect to moduli $(\tau,\,\mu)$. In particular, to check its splittability in τ and μ. This fact could be also checked, due to the heat equation, by investigating closeness of differentiations of sixteen functions $\Theta\big[{\alpha \atop \beta}\big](z_1, z_2)$ with respect to arguments (z_1, z_2). As it is seen from this example, the closure does exist and it being algebraic at the least. Whether it will be polynomial and, if no, how to do this?

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